Worksheet 1.7 Further Signed Numbers

Section 1 Multiplication of signed numbers

Multiplication is a shorthand way of adding together a large number of the same thing. For example, if I have 3 bags of oranges with 5 oranges in each bag, then I have 15 oranges in total: $5 + 5 + 5 = 5 \times 3 = 15$. Notice that if instead I had 5 bags of oranges with 3 in each bag, then I would still have 15 oranges: $3 + 3 + 3 + 3 + 3 = 3 \times 5 = 15$.

This illustrates a mathematical property called commutitivity which says that the order of multiplication doesn't matter: $a \times b = b \times a$. All real numbers have this property. Notice that

$$-1 \times 5 = 5 \times -1 = -5$$

Because I can arrange numbers in any order when I multiply numbers that do not have the same sign, I can write

$$\begin{aligned} -5 \times 4 &= -1 \times 5 \times 4 \\ &= -1 \times 20 \\ &= -20 \end{aligned}$$

Similarly,

$$\begin{aligned} 5 \times -4 &= 5 \times 4 \times -1 \\ &= 20 \times -1 \\ &= -20 \end{aligned}$$

Now, to be able to multiply any two numbers of any sign, we need one more piece of information, and that is:

$$-1 \times -1 = 1$$

Example 1 :

$$\begin{aligned} -5 \times -4 &= -1 \times 5 \times -1 \times 4 \\ &= -1 \times -1 \times 5 \times 4 \\ &= -1 \times -1 \times 20 \\ &= 1 \times 20 \\ &= 20 \end{aligned}$$

With a little thought we can come up with the following rules:

- multiplying numbers with like signs gives a positive number
- multiplying numbers with unlike signs gives a negative number

Exercises:

1. Perform the following multiplications:

   (a) -3×4

   (b) 7×-2

   (c) -16×-1

   (d) 8×4

   (e) -9×-4

   (f) $-12 \times \frac{1}{2}$

   (g) $-\frac{1}{4} \times -\frac{1}{2}$

   (h) $-27 \times -\frac{1}{3}$

   (i) -1.2×3

   (j) -1.47×-10

Section 2 Division of signed numbers

If I had a box of 60 apples, and I wanted to divide it evenly amongst 4 people, each person would get 15 apples. The mathematical formula that expresses this calculation is:

$$60 \div 4 = 15$$

Another way of looking at the apple situation is that I need to find one quarter of 60. The formula would then be written:

$$60 \times \frac{1}{4} = 15$$

These two formulae express the same thing. For any two numbers x and y (so long as $y \neq 0$),

$$x \div y = x \times \frac{1}{y}$$

Dividing by a number is the same thing as multiplying by the reciprocal of the number. The reciprocal of y is $\frac{1}{y} = 1 \div y$ for $y \neq 0$. This is true even if y is a fraction, or is negative (or both). Now we can deal with division of signed numbers in a similar way to the multiplication of signed numbers.

Example 2 :

$$\begin{aligned} 60 \div -4 &= 60 \times \frac{-1}{4} \\ &= 60 \times \frac{1}{4} \times -1 \\ &= 15 \times -1 \\ &= -15 \end{aligned}$$

Example 3 :

$$\begin{aligned} -60 \div -3 &= -60 \times \frac{-1}{3} \\ &= -1 \times 60 \times \frac{1}{3} \times -1 \\ &= -1 \times -1 \times 20 \\ &= 1 \times 20 \\ &= 20 \end{aligned}$$

Notice that the last expression could have been written:

$$\frac{-60}{-3} = \frac{-1}{-1} \times \frac{60}{3} = 1 \times 20$$

since the division of one number by itself is always 1.

An important thing to notice with signed fractions is that all of the following expressions are equivalent:

$$\frac{-6}{7} = -1 \times \frac{6}{7} = \frac{6}{-7}$$

The usual way to write this is with the minus sign on top, or sitting out in front of the entire fraction as in

$$\frac{-a}{b} = -\frac{a}{b}$$

As with multiplication, we have a rule for dividing signed numbers:

- dividing numbers with like signs gives a positive number
- dividing numbers with unlike signs gives a negative number

When doing divisions, it is important to keep in mind that division by zero is not defined.

Exercises:

1. Perform the following divisions:

   (a) $-8 \div -2$

   (b) $-20 \div -5$

   (c) $6 \div 3$

   (d) $18 \div 9$

   (e) $-36 \div -4$

   (f) $8 \div -\frac{1}{2}$

   (g) $-\frac{5}{8} \div -\frac{1}{4}$

   (h) $16 \div -\frac{1}{8}$

   (i) $-\frac{2}{5} \div \frac{7}{10}$

   (j) $2\frac{1}{4} \div -9$

Section 3 More multiplication and division

When confronted by a multiplication and/or division which has more than two numbers involved, we just extend the processes already described. Here's a couple of examples:

Example 1 :

$$
\begin{aligned}
-2 \times -3 \times 5 \times -2 &= -1 \times 2 \times -1 \times 3 \times 5 \times -1 \times 2 \\
&= -1 \times -1 \times -1 \times 2 \times 3 \times 5 \times 2 \\
&= 1 \times -1 \times 30 \times 2 \\
&= -1 \times 60 \\
&= -60
\end{aligned}
$$

Alternatively, this may be done working from left to right:

$$
\begin{aligned}
-2 \times -3 \times 5 \times -2 &= (-2 \times -3) \times 5 \times -2 \\
&= 6 \times 5 \times -2 \\
&= 30 \times -2 \\
&= -60
\end{aligned}
$$

Example 2 :

$$\begin{aligned} -2 \times -5 \times 6 \times 3 &= -1 \times 2 \times -1 \times 5 \times 6 \times 3 \\ &= -1 \times -1 \times 2 \times 5 \times 6 \times 3 \\ &= 1 \times 10 \times 18 \\ &= 180 \end{aligned}$$

This example may also be done working from left to right:

$$\begin{aligned} -2 \times -5 \times 6 \times 3 &= (-2 \times -5) \times 6 \times 3 \\ &= 10 \times 6 \times 3 \\ &= 60 \times 3 \\ &= 180 \end{aligned}$$

Notice that because $-1 \times -1 = 1$, when there are an even number of negative numbers the product will be positive and when there are an odd number of negative numbers, the product will be negative. This results holds for division as well, because we can rewrite any division as multiplication by the reciprocal.

Exercises:

1. Evaluate

   (a) $-3 \times 4 \times -5$
   (b) $6 \times -2 \times 3$
   (c) $-20 \times -4 \times -5$
   (d) $-2 \times -3 \times 2 \times -4$
   (e) $\frac{1}{2} \times 30 \times -2$
   (f) $40 \times -\frac{1}{2} \times -\frac{1}{4} \times 2$
   (g) $18 \div -\frac{1}{5}$
   (h) $-20 \div 2 \times \frac{1}{4}$
   (i) $800 \times -\frac{1}{20} \times -3 \times -\frac{1}{10}$
   (j) $(6 \times -\frac{1}{2}) \times (20 \div \frac{1}{4})$
   (k) $(30 \div -2) \div (-10 \div -2)$
   (l) $2\frac{1}{2} \times \frac{2}{7} \times -\frac{1}{5}$
   (m) $-8\frac{1}{4} \div -3 \times -\frac{2}{5}$

(n) $-\frac{3}{4} \times -\frac{2}{9} \times -4 \times -5$

(o) $16 \times \frac{1}{2} \times 3 \times 5$

(p) $\frac{2}{3} \times -1 \times 8 \div -2$

(q) $-1 \times 7 \times -3 \times -\frac{1}{4} \times 20$

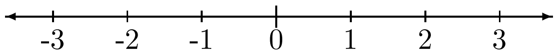
(r) $(-6 \times 4) \div (-8 \times \frac{1}{2})$

(s) $(20 \div -2) \times (-18 \div -9)$

(t) $(-1)^2 \times (-2)^3$

Section 4 Introduction to absolute values

When we write $|a|$, we are referring to the absolute value of a. We can think of the absolute value of a number as its distance away from zero. So if we look at the number line,



then we see that -3 is 3 units away from zero. So $|-3| = 3$. Absolute values are never negative.

Example 1 :

$$\begin{aligned} |-7| &= 7 \\ |6| &= 6 \\ |7| &= 7 \end{aligned}$$

If we are asked to work out the absolute value of a sum such as $|-a+b|$, we should treat the vertical bars as brackets in the sense that we do the operations inside the bars first. So

$$\begin{aligned} |-3+2| &= |-1| = 1 \\ |-5 \times 2 + 6| &= |-10+6| = |-4| = 4 \end{aligned}$$

Notice that we cannot generally write $|a+b| = |a|+|b|$. Try some a's and b's to satisfy yourself of this. Try making one of a or b negative.

Example 2 : If I went for a Sunday drive and went 30 kilometres east and then 15 kilometres west, I would be 15 kilometres from my starting point even though I had traveled 45 kilometres in total. This illustrates that we cannot take the absolute value of the inside bits separately. If we think of east as a positive direction, and west as a negative direction, then we have been considering the problem

$$\begin{aligned} |\text{Thirty km east} + \text{Fifteen km west}| &= |30 - 15| \\ &= |15| \\ &= 15 \end{aligned}$$

On the other hand, if we mistakenly consider the quantity as the absolute value of each bit:

$$\begin{aligned} |\text{Thirty km east}| + |\text{ Fifteen km west}| &= |30| + |-15| \\ &= 30 + 15 \\ &= 45 \end{aligned}$$

Therefore we can see that

$$|30 - 15| \neq |30| + |-15|$$

Exercises 1.7 Further Signed Numbers

1. (a) Which of the following have the same answer?

   i. 6×3
   ii. $+6 \times (-3)$
   iii. $-6 \times (+3)$
   iv. $-6 \times (-3)$
   v. $(+6) \times (+3)$
   vi. -6×3

   (b) Which are equivalent?

   i. $-6 \div +7$
   ii. $\frac{-6}{7}$
   iii. $\frac{6}{7}$
   iv. $-\frac{6}{7}$
   v. $\frac{-6}{-7}$
   vi. $-6 \div -7$

   (c) Evaluate the following:

   i. $30 \times (-2)$
   ii. $-25 \times (-4)$
   iii. $30 \div (-2)$
   iv. $(-32) \div (-8)$
   v. $-100 \div 25$
   vi. $(-2) \times 4 \times (-5)$
   vii. $(-6) \times (-3) \times (-7)$
   viii. $-6 \times (-3) + (2) \times 4$
   ix. $-5 + 4 \times (-7)$
   x. $-9 \div (-3) + 7 \times (-2)$
   xi. $6(-2) + 4(-3)$
   xii. $-2(-7) - (-3)(-4)$

   (d) Evaluate the following:

   i. $|-4|$
   ii. $|-2| + |3|$
   iii. $|-2 + 3|$
   iv. $|4| - |-4|$
   v. $|4 - (-4)|$
   vi. $|1 - |3||$

Answers 1.7

Section 1

1. (a) -12
   (b) -14
   (c) 16
   (d) 32
   (e) 36
   (f) -6
   (g) $\frac{1}{8}$
   (h) 9
   (i) -3.6
   (j) 14.7

Section 2

1. (a) 4
   (b) 4
   (c) 2
   (d) 2
   (e) 9
   (f) -16
   (g) $\frac{5}{2}$
   (h) -128
   (i) $-\frac{4}{7}$
   (j) $-\frac{1}{4}$

Section 3

1. (a) 60
   (b) -36
   (c) -400
   (d) -48
   (e) -30
   (f) 10
   (g) -90
   (h) $-\frac{5}{2}$
   (i) -12
   (j) -240
   (k) -3
   (l) $-\frac{1}{7}$
   (m) $-\frac{11}{10}$
   (n) $\frac{10}{3}$
   (o) 120
   (p) $\frac{8}{3}$
   (q) -105
   (r) 6
   (s) -20
   (t) -8

Exercises 1.7

1. (a) i,iv and v are the same. ii, iii and vi are the same.
   (b) i, ii and iv are equivalent. iii, v and vi are equivalent.
   (c) i. -60
   ii. 100
   iii. -15
   iv. 4
   v. -4
   vi. 40
   vii. -126
   viii. 26
   ix. -33
   x. -11
   xi. -24
   xii. 2
   (d) i. 4
   ii. 5
   iii. 1
   iv. 0
   v. 8
   vi. 2